Chapter 4
Infinite series

4.1 Introduction

A lot of students are confused between sequences and series, so we have first to clarify the difference between them.

Sequence is a function whose domain is the set of positive integers, the sequence $\{S_n\}_{n=1}^{\infty}$ converge to a limit if

$$\lim_{n\to\infty} S_n = L\,,\ L \in \Re \tag{1}$$

A series is the sum of the terms of a sequence. Finite sequences and series have defined first and last terms, whereas infinite sequences and series continue indefinitely. In mathematics, given an infinite sequence of numbers $\{\ a_n\ \}$, a series is informally the result of adding all those terms together: $a_1 + a_2 + a_3 + \cdot\cdot\cdot$

These can be written more compactly using the summation symbol $\sum$.

Series is summation of number of terms, if the number of terms is finite, then it is called finite series, but if the number of terms is infinite, then it is called infinite series.

A necessary but not sufficient condition for the series $\sum_{n=1}^{\infty} U_n$ to be convergent is $\lim_{n\to\infty} U_n = 0$.

Example 1

Which of the following series may be convergent?

$$\sum_{n=1}^{\infty}\frac{n}{n+1} \qquad \sum_{n=1}^{\infty}\frac{1}{n+1} \qquad \sum_{n=1}^{\infty}\frac{3^n}{n+1}$$

Solution:

The first and third series are divergent since $\lim_{n\to\infty}\frac{n}{n+1} \neq 0$ and $\lim_{n\to\infty}\frac{3^n}{n+1} \neq 0$, while the second series may be convergent since $\lim_{n\to\infty}\frac{1}{n+1} = 0$, but by test of convergence we will find that this series is divergent.

Special cases

1-Sequence $\{r^n\}_{n=1}^{\infty}$ is convergent if $\lim_{n\to\infty} r^n = 0$, $-1 < r < 1$

2- Sequence $\{r^n\}_{n=1}^{\infty}$ is divergent if $\lim_{n\to\infty} r^n = \infty$, $r > 1$ or $r < -1$

3- Series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $P > 1$

4- Series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent if $P \le 1$

5- Series $\sum_{n=1}^{\infty} P^n$ is convergent if $0 < P < 1$

6- Series $\sum_{n=1}^{\infty} P^n$ is divergent if $P \ge 1$

Example 2

Test the following series for convergence:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^3, \qquad \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n, \qquad \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^{1-n}, \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$$

Solution:

Refer to the above special cases 3, 4, 5, 6, we will get the first and second series are convergent while the third and fourth series are divergent.

Cauchy sequence

Every convergent sequence is called Cauchy sequence

Example 3

Determine Cauchy sequence from the following sequences:

$$\left\{3^n\right\}_{n=1}^{\infty}, \left\{\frac{n}{3^n}\right\}_{n=1}^{\infty}, \left\{\frac{1}{n}\right\}_{n=1}^{\infty}, \left\{\frac{3n}{n+1}\right\}_{n=1}^{\infty}, \left\{\frac{n^3}{n^2+1}\right\}_{n=1}^{\infty}, \left\{\left(\frac{n+3}{n}\right)^{2n}\right\}_{n=1}^{\infty}$$

Solution:

Refer to second point in the above special cases, therefore $\left\{3^n\right\}_{n=1}^{\infty}$ is divergent and refer to formula (1) for determining whether the other sequences convergent or not, therefore

$\left\{\frac{n}{3^n}\right\}_{n=1}^{\infty}$, $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$, $\left\{\frac{3n}{n+1}\right\}_{n=1}^{\infty}$, $\left\{(\frac{n+3}{n})^{2n}\right\}_{n=1}^{\infty}$ are convergent sequences, since $\lim_{n\to\infty}\frac{n}{3^n}=0$, $\lim_{n\to\infty}\frac{1}{n}=0$, $\lim_{n\to\infty}\frac{3n}{n+1}=3$, $\lim_{n\to\infty}(\frac{n+3}{n})^{2n}=e^6$, but $\left\{\frac{n^3}{n^2+1}\right\}_{n=1}^{\infty}$ is divergent sequence since $\lim_{n\to\infty}\frac{n^3}{n^2+1}=\infty$, hence the convergent sequences are Cauchy sequences.

Report 1

Is the sequence $\left\{\frac{\cos n}{3^n}\right\}_{n=1}^{\infty}$ convergent?

4.2 Test of Convergence

As we know, the necessary but not sufficient condition for the series $\sum_{n=1}^{\infty}U_n$ to be convergent is $\lim_{n\to\infty}U_n=0$, and then we have to use suitable method for test of convergence from the following methods.

A- Cauchy test (n^{th} root test)

Consider the series $\sum_{n=1}^{\infty} U_n$, and then to test whether the series is convergent or divergent using Cauchy test, apply the following steps:

- Take the n^{th} root for U_n, then take the limit of $\sqrt[n]{U_n}$ when n tends to infinity,
- If $\lim_{n\to\infty} \sqrt[n]{U_n} < 1$, then $\sum_{n=1}^{\infty} U_n$ is convergent, if $\lim_{n\to\infty} \sqrt[n]{U_n} > 1$, then $\sum_{n=1}^{\infty} U_n$ is divergent, but if $\lim_{n\to\infty} \sqrt[n]{U_n} = 1$, then this test fail and we have to search for another method.

Example 4

Test the following series for convergence:

a) $\sum_{n=1}^{\infty} \frac{n^n}{3^n}$, b) $\sum_{n=1}^{\infty} e^{-n^2}$, c) $\sum_{n=1}^{\infty} \frac{1}{n^n}$, d) $\sum_{n=1}^{\infty} (\frac{n+3}{n})^{2n^2}$,

e) $\sum_{n=1}^{\infty} (\frac{n}{2n+3})^n$.

Solution:

a) Since $\lim\limits_{n\to\infty}\sqrt[n]{\dfrac{n^n}{3^n}} = \lim\limits_{n\to\infty}\dfrac{n}{3} = \infty > 1$, therefore $\sum\limits_{n=1}^{\infty}\dfrac{n^n}{3^n}$ is divergent,

b) Since $\lim\limits_{n\to\infty}\sqrt[n]{e^{-n^2}} = \lim\limits_{n\to\infty} e^{-n} = 0 < 1$, therefore $\sum\limits_{n=1}^{\infty} e^{-n^2}$ is convergent,

c) Since $\lim\limits_{n\to\infty}\sqrt[n]{\dfrac{1}{n^n}} = \lim\limits_{n\to\infty}\dfrac{1}{n} = 0 < 1$, therefore $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^n}$ is convergent,

d) Since $\lim\limits_{n\to\infty}\sqrt[n]{\left(\dfrac{n+3}{n}\right)^{2n^2}} = \lim\limits_{n\to\infty}\left(\dfrac{n+3}{n}\right)^{2n} = e^6 > 1$, thus $\sum\limits_{n=1}^{\infty}\left(\dfrac{n+3}{n}\right)^{2n^2}$ is divergent,

e) Since $\lim\limits_{n\to\infty}\sqrt[n]{\left(\dfrac{n}{2n+3}\right)^{n}} = \lim\limits_{n\to\infty}\left(\dfrac{n}{2n+3}\right) = \dfrac{1}{2} < 1$, hence $\sum\limits_{n=1}^{\infty}\left(\dfrac{n}{2n+3}\right)^{n}$ is convergent.

B- Integral test

Consider the series $\sum_{n=1}^{\infty} U_n$, then to test whether the series is convergent or divergent using integral test so that if $\int_1^{\infty} U_n \, dn$ equal any real number, then $\sum_{n=1}^{\infty} U_n$ is convergent and if $\int_1^{\infty} U_n \, dn = \infty$, then $\sum_{n=1}^{\infty} U_n$ is divergent.

Example 5

Test the following series for convergence:

a) $\sum_{n=1}^{\infty} n\, e^{-n^2}$, b) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$, c) $\sum_{n=1}^{\infty} \frac{e^n}{e^{2n}+1}$, d) $\sum_{n=1}^{\infty} \frac{1}{e^{2n}+1}$,

e) $\sum_{n=1}^{\infty} \frac{1}{n}$.

Solution:

a) Since $\int_1^{\infty} n\, e^{-n^2} \, dn = \frac{-1}{2}\int_1^{\infty} -2n\, e^{-n^2} \, dn = \frac{-1}{2}(e^{-n^2})_1^{\infty} = \frac{1}{2e}$, thus $\sum_{n=1}^{\infty} n\, e^{-n^2}$ is convergent,

b) Since $\int_1^\infty \frac{1}{n^2+1}\,dn = (\tan^{-1} n)_1^\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$, therefore $\sum_{n=1}^\infty \frac{1}{n^2+1}$ is convergent,

c)Since $\int_1^\infty \frac{e^n}{e^{2n}+1}\,dn = (\tan^{-1} e^n)_1^\infty = \tan^{-1}\infty - \tan^{-1} e = \frac{\pi}{2} - \tan^{-1} e$,

therefore $\sum_{n=1}^\infty \frac{e^n}{e^{2n}+1}$ is convergent,

d) $\int_1^\infty \frac{1}{e^{2n}+1}\,dn = \int_1^\infty \left(\frac{1+e^{2n}-e^{2n}}{e^{2n}+1}\right) dn = \int_1^\infty \left(1 - \frac{e^{2n}}{e^{2n}+1}\right) dn$

$= \left(n - \frac{\ln(e^{2n}+1)}{2}\right)_1^\infty = \frac{\ln(e^2+1)}{2} - 1$, therefore $\sum_{n=1}^\infty \frac{1}{e^{2n}+1}$ is convergent,

e) Since $\int_1^\infty \frac{1}{n}\,dn = (\ln n)_1^\infty = \infty$, therefore $\sum_{n=1}^\infty \frac{1}{n}$ is divergent.

C - Ratio test

Consider the series $\sum_{n=1}^{\infty} U_n$, and then to test whether the series is convergent or divergent using ratio test, apply the following steps:

- Find U_{n+1} and obtain the ratio $\frac{U_{n+1}}{U_n}$,
- Take the limit of the ratio $\frac{U_{n+1}}{U_n}$ when n tends to infinity,
- If $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} < 1$, then $\sum_{n=1}^{\infty} U_n$ is convergent and if

$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} > 1$, then $\sum_{n=1}^{\infty} U_n$ is divergent, but if $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = 1$, then this test fail and we have to search for another method.

Example 6

Test the following series for convergence:

a) $\sum_{n=1}^{\infty} \frac{1}{n!}$, b) $\sum_{n=1}^{\infty} \frac{3^n}{n^2+1}$, c) $\sum_{n=1}^{\infty} \frac{2n!}{(n!)^2}$, d) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$,

e) $\sum_{n=1}^{\infty} \frac{n}{e^n}$.

Solution:

a) Since $U_{n+1} = \dfrac{1}{(n+1)!}$, hence the ratio $\dfrac{U_{n+1}}{U_n} = \dfrac{n!}{(n+1)!} = \dfrac{1}{n+1}$,

so $\lim\limits_{n\to\infty} \dfrac{U_{n+1}}{U_n} = 0 < 1$, therefore $\sum\limits_{n=1}^{\infty} \dfrac{1}{n!}$ is convergent,

b) Since $U_{n+1} = \dfrac{3^{n+1}}{(n+1)^2+1}$, hence the ratio

$\dfrac{U_{n+1}}{U_n} = \dfrac{3^{n+1}(n^2+1)}{((n+1)^2+1)3^n} = \dfrac{3(n^2+1)}{(n+1)^2+1}$, so $\lim\limits_{n\to\infty} \dfrac{U_{n+1}}{U_n} = 3 > 1$ and

thus $\sum\limits_{n=1}^{\infty} \dfrac{3^n}{n^2+1}$ is divergent,

c) Since $U_{n+1} = \dfrac{(2n+2)!}{((n+1)!)^2}$, hence the ratio

$\dfrac{U_{n+1}}{U_n} = \dfrac{(2n+2)!(n!)^2}{((n+1)!)^2 2n!} = \dfrac{(2n+2)(2n+1)}{(n+1)^2}$, so $\lim\limits_{n\to\infty} \dfrac{U_{n+1}}{U_n} = 4 > 1$,

therefore $\sum\limits_{n=1}^{\infty} \dfrac{2n!}{(n!)^2}$ is divergent,

d) Since $U_{n+1} = \dfrac{(n+1)^{n+1}}{(n+1)!}$, hence the ratio

$$\frac{U_{n+1}}{U_n} = \frac{(n+1)^{n+1} n!}{(n+1)! n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n, \quad \text{so} \quad \lim_{n\to\infty} \frac{U_{n+1}}{U_n} = e > 1,$$

therefore $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ is divergent,

e) Since $U_{n+1} = \dfrac{n+1}{e^{n+1}}$, hence the ratio $\dfrac{U_{n+1}}{U_n} = \dfrac{(n+1)e^n}{n\, e^{n+1}} = \dfrac{n+1}{ne}$,

so $\displaystyle\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{1}{e} < 1$, therefore $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$ is convergent.

D- Comparison test

Consider the series $\sum_{n=1}^{\infty} U_n$, then to test whether the series is convergent or divergent using comparison test. This done by Choosing series $\sum_{n=1}^{\infty} V_n$ from $\sum_{n=1}^{\infty} U_n$ so that if $V_n \geq U_n$ and V_n convergent, then U_n convergent and if $V_n \leq U_n$ and V_n divergent, then U_n divergent.

Example 7

Test the following series for convergence:

a) $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt[5]{n^2+1}}$, b) $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$, c) $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$, d) $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^5+n}}$,

e) $\sum_{n=1}^{\infty} \frac{1}{(2n+3)^3}$.

Solution:

a) The series $\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} n^{8/5}$ is divergent, therefore $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt[5]{n^2+1}}$ is divergent as $n^{8/5} = \frac{n^2}{\sqrt[5]{n^2+1}}$ as n tends to infinity.

b) The series $\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent, therefore $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ is convergent as $\frac{1}{n^3} > \frac{1}{n^3+1}$.

c) The series $\sum_{n=1}^{\infty} V_n = \sum_{n=2}^{\infty} \frac{1}{n}$ is divergent, therefore $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ is divergent as $\frac{1}{n} < \frac{1}{\ln(n)}$.

d) The series $\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent, therefore $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^5+n}}$ is divergent as $\frac{1}{\sqrt{n}} = \frac{n^2}{\sqrt{n^5+n}}$ as n tends to infinity.

e) The series $\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} \frac{1}{8n^3}$ is convergent, therefore $\sum_{n=1}^{\infty} \frac{1}{(2n+3)^3}$ is convergent as $\frac{1}{n^3} > \frac{1}{(2n+3)^3}$.

Note: For the series $\sum_{n=1}^{\infty} U_n$, if $\lim_{n\to\infty} U_n \neq 0$, then $\sum_{n=1}^{\infty} U_n$ is divergent

4.3 Alternating series

In mathematics, an alternating series is an infinite series of the form $\sum_{n=1}^{\infty}(-1)^n U_n$ with $U_n \geq 0$ (or $U_n \leq 0$) for all n. A finite sum of this kind is an alternating sum. An alternating series converges if the terms U_n converge to 0 monotonically. It may be absolutely convergent or conditionally convergent or divergent.

The alternating series $\sum_{n=1}^{\infty}(-1)^n U_n$ to be absolutely convergent or conditionally convergent must satisfy the following conditions:

a) $\lim_{n\to\infty} U_n = 0$,   b) $U_n > U_{n+1}$, otherwise it will be divergent.

If the above conditions are satisfied and $\sum_{n=1}^{\infty} U_n$ is convergent, then $\sum_{n=1}^{\infty}(-1)^n U_n$ is called absolutely convergent and if

$\sum_{n=1}^{\infty} U_n$ is divergent, then $\sum_{n=1}^{\infty} (-1)^n U_n$ is called conditionally convergent.

Example 8

Determine the convergent series and classify them

a) $\sum_{n=1}^{\infty} \frac{2n(-1)^{n-1}}{4n^2-3}$, b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+3}$, c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$,

d) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n)}$, e) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 3^n}{n!}$.

Solution

a) Let $U_n = \frac{2n}{4n^2-3}$, $\lim_{n\to\infty} \frac{2n}{4n^2-3} = 0$, $U_{n+1} = \frac{2n+2}{4(n+1)^2-3}$, hence $U_n > U_{n+1}$. By using integral test, we will get that $\sum_{n=1}^{\infty} \frac{2n}{4n^2-3}$ is divergent, so $\sum_{n=1}^{\infty} \frac{2n(-1)^{n-1}}{4n^2-3}$ is called conditionally convergent.

b) Let $U_n = \frac{1}{n^2+3}$, $\lim\limits_{n\to\infty} \frac{1}{n^2+3} = 0$, $U_{n+1} = \frac{1}{(n+1)^2+3}$, hence $U_n > U_{n+1}$. By using integral test, we will get that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2+3}$ is convergent, so $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+3}$ is called absolutely convergent.

c) Let $U_n = \frac{1}{n}$, $\lim\limits_{n\to\infty} \frac{2n}{4n^2-3} = 0$, $U_{n+1} = \frac{1}{n+1}$, so $U_n > U_{n+1}$. By using integral test, we will get that $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is divergent, and then $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is called conditionally convergent.

d) Let $U_n = \frac{1}{\ln(n)}$, $\lim\limits_{n\to\infty} \frac{1}{\ln(n)} = 0$, $U_{n+1} = \frac{1}{\ln(n+1)}$, hence $U_n > U_{n+1}$. By using comparison test, we will get that $\sum\limits_{n=1}^{\infty} \frac{1}{\ln(n)}$ is divergent, so $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n)}$ is called conditionally convergent.

e) Let $U_n = \frac{3^n}{n!}$, $\lim_{n\to\infty} \frac{3^n}{n!} = 0$, $U_{n+1} = \frac{3^{n+1}}{(n+!)!}$, hence $U_n > U_{n+1}$.

By using ratio test, we will get that $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ is convergent, so $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 3^n}{n!}$ is called absolutely convergent.

4.4 Power series

In mathematics, a power series (in one variable) is an infinite series of the form $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots$, where a_n represents the coefficient of the nth term, c is a constant, and x varies around c (for this reason one sometimes speaks of the series as being centered at c). This series usually arises as the Taylor series of some known function; the Taylor series article contains many examples. In many situations c is equal to zero, for instance when considering a Maclaurin series. In such cases, the power series takes the simpler form:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots.$$

4.4.1 Interval of convergence of power series

To determine the interval of convergence of power series $\sum_{n=1}^{\infty} u_n (x^b)^n$, we have to use ratio test such that $\lim_{n\to\infty} \left| \frac{u_{n+1} (x^b)^{n+1}}{u_n (x^b)^n} \right| < 1$ is the sufficient and necessary condition to obtain the interval of convergence.

Example 9

Find interval of convergence for the following power series

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n^2+3}$, b) $\sum_{n=1}^{\infty} \frac{3^n x^{n-1}}{n!}$, c) $\sum_{n=1}^{\infty} \frac{n e^{-n^2}}{x^n}$, d) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^3+1}$,

e) $\sum_{n=1}^{\infty} \frac{3^n}{(n^2+1)(x-2)^n}$.

Solution

a) Since $U_n = \dfrac{(-1)^{n-1} x^{2n}}{n^2+3}$, and $U_{n+1} = \dfrac{(-1)^{n} x^{2n+2}}{(n+1)^2+3}$, hence the ratio $\left|\dfrac{U_{n+1}}{U_n}\right| = \left|\dfrac{n^2+3}{(n+1)^2+3}\dfrac{(-1)^{n} x^{2n+2}}{(-1)^{n-1} x^{2n}}\right| = \left|-\dfrac{(n^2+3)x^2}{(n+1)^2+3}\right|$, therefore $\lim\limits_{n\to\infty}\left|\dfrac{U_{n+1}}{U_n}\right| = \lim\limits_{n\to\infty}\left|-\dfrac{(n^2+3)x^2}{(n+1)^2+3}\right| = \lim\limits_{n\to\infty}\left|x^2\right| < 1$ to be convergent, hence $-1 < x < 1$ is the interval of convergence.

b) Since $U_n = \dfrac{3^n x^{n-1}}{n!}$, and $U_{n+1} = \dfrac{3^{n+1} x^{n}}{(n+1)!}$, hence the ratio $\left|\dfrac{U_{n+1}}{U_n}\right| = \left|\dfrac{n!3^{n+1} x^{n}}{(n+1)!3^n x^{n-1}}\right| = \left|\dfrac{3x}{(n+1)}\right|$, thus $\lim\limits_{n\to\infty}\left|\dfrac{U_{n+1}}{U_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{3x}{(n+1)}\right|$ $= 0 < 1$, hence $\sum\limits_{n=1}^{\infty}\dfrac{3^n x^{n-1}}{n!}$ is convergent for all x.

c) Since $U_n = \dfrac{n\,e^{-n^2}}{x^n}$, and $U_{n+1} = \dfrac{(n+1)\,e^{-(n+1)^2}}{x^{n+1}}$, hence the ratio $\left|\dfrac{U_{n+1}}{U_n}\right| = \left|\dfrac{x^n(n+1)\,e^{-(n+1)^2}}{x^{n+1}\;\; n\,e^{-n^2}}\right| = \left|\dfrac{(n+1)}{ne^{(2n+1)}x}\right|$, thus $\lim\limits_{n\to\infty}\left|\dfrac{U_{n+1}}{U_n}\right|$

$= \lim\limits_{n\to\infty}\left|\dfrac{(n+1)}{ne^{(2n+1)}x}\right| = 0 < 1$, so $\sum\limits_{n=1}^{\infty}\dfrac{n\,e^{-n^2}}{x^n}$ is convergent for all x.

d) Since $U_n = \dfrac{(x-2)^n}{n^3+1}$, and $U_{n+1} = \dfrac{(x-2)^{n+1}}{(n+1)^3+1}$, hence the ratio

$\left|\dfrac{U_{n+1}}{U_n}\right| = \left|\dfrac{(n^3+1)(x-2)^{n+1}}{((n+1)^3+1)(x-2)^n}\right| = \left|\dfrac{(n^3+1)(x-2)}{((n+1)^3+1)}\right|$, therefore

$\lim\limits_{n\to\infty}\left|\dfrac{U_{n+1}}{U_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n^3+1)(x-2)}{((n+1)^3+1)}\right| = \lim\limits_{n\to\infty}|x-2| < 1$ to be convergent, hence $1 < x < 3$ is the interval of convergence.

e) Since $U_n = \dfrac{3^n}{(n^2+1)(x-2)^n}$, & $U_{n+1} = \dfrac{3^{n+1}}{((n+1)^2+1)(x-2)^{n+1}}$,

hence the ratio :

$$\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{3^{n+1}(n^2+1)(x-2)^n}{3^n((n+1)^2+1)(x-2)^{n+1}}\right| = \left|\frac{3(n^2+1)}{((n+1)^2+1)(x-2)}\right|$$

Therefore

$$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{3(n^2+1)}{((n+1)^2+1)(x-2)}\right| = \lim_{n\to\infty}\left|\frac{3}{(x-2)}\right| < 1$$ to be convergent, hence $|x-2| > 3$, thus x > 5 or x < -1 the interval of convergence.

4.5 Problems

I) Do the following sequences $\{a_n\}$ converge or diverge?

a) $a_n = \dfrac{n}{e^n}$, b) $a_n = \left[1+\dfrac{2}{n}\right]^n$, c) $a_n = \dfrac{(-1)^{n-1} n}{2^{n-1}}$

d) $a_n = \dfrac{2n-7}{3n+1}$, e) $a_n = \dfrac{n^2}{2^n}$, f) $a_n = \sqrt{n+2} - \sqrt{n+1}$

II) Test the following series for convergence

a) $\sum_{n=1}^{\infty}\frac{n}{4^{2n+1}(n+1)}$ $\sum_{n=1}^{\infty}\frac{n^2}{(3n+1)!}$ $\sum_{n=1}^{\infty}\frac{4^n+9^n}{5^n+11^n}$

$\sum_{n=1}^{\infty}\frac{1}{7^n+5n}$ $\sum_{n=1}^{\infty}\frac{3^n+2}{5^n+7^n}$ $\sum_{n=1}^{\infty}\frac{n}{3^n(2n!)^2}$ $\sum_{n=1}^{\infty}\frac{n!}{3^{(1+2n)}}$

$\sum_{n=1}^{\infty}\frac{3^n}{(2n+1)!}$ $\sum_{n=1}^{\infty}\frac{n!}{n\,2^n}$ $\sum_{n=1}^{\infty}\frac{7^n+2}{9^n}$

b) $\sum_{n=1}^{\infty}\frac{n^n}{3^{(1+2n)}}$ $\sum_{n=1}^{\infty}[\frac{2n^2+1}{n^2+1}]^n$ $\sum_{n=1}^{\infty}[\frac{5n^2-3n^3}{7n^3+2}]^{5n}$

$\sum_{n=1}^{\infty}\frac{3^n}{5^n}$ $\sum_{n=1}^{\infty}\frac{n^n}{(3n+1)^n}$ $\sum_{n=1}^{\infty}e^{-n^2}$ $\sum_{n=1}^{\infty}\frac{e^n}{n^n}$

$\sum_{n=1}^{\infty}(\frac{2n+3}{n})^{4n}$ $\sum_{n=1}^{\infty}(\frac{n}{2n+3})^n$ $\sum_{n=1}^{\infty}\frac{3^n}{((2n+1)!)^n}$

c) $\sum_{n=2}^{\infty}\frac{Ln(n)}{n}$ $\sum_{n=1}^{\infty} n\, e^{n^2}$ $\sum_{n=2}^{\infty}\frac{3}{n\ln(n)}$

$\sum_{n=1}^{\infty}\frac{1}{e^n+1}$ $\sum_{n=1}^{\infty}\frac{\cos n+\sin n}{\sin n-\cos n}$ $\sum_{n=1}^{\infty}\frac{n}{4n^2-3}$ $\sum_{n=1}^{\infty}\frac{1}{n^2+9}$

$\sum_{n=1}^{\infty}\frac{7}{3n^2+2n}$ $\sum_{n=1}^{\infty}\frac{1}{3n}$ $\sum_{n=1}^{\infty}\frac{n}{2n+1}$

d) $\sum_{n=1}^{\infty}\frac{4n^2+3n}{\sqrt[7]{n^5+n^3}}$ $\sum_{n=1}^{\infty}\frac{5n-2}{9n+7}$ $\sum_{n=1}^{\infty}\frac{3n^2+n}{n^4+5}$

$\sum_{n=2}^{\infty}\frac{1}{Ln(n)}$ $\sum_{n=1}^{\infty}\frac{7}{3n^2+2n}$ $\sum_{n=1}^{\infty}\frac{3n^2+n}{n^4+5}$ $\sum_{n=1}^{\infty}\frac{\sqrt[5]{n^4+n}}{n^2+5}$

$\sum_{n=1}^{\infty}\frac{n^3+n}{\sqrt[5]{n^4+1}}$ $\sum_{n=1}^{\infty}\frac{1}{n^4+n}$ $\sum_{n=1}^{\infty}\frac{1}{(2n^2+3)^4}$

e) $\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n(2n!)^2}$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{(n^2+9)}$ $\sum_{n=0}^{\infty}\frac{(-1)^{n-3}\sqrt{n}}{n+4}$

$\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^2+1}$ $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n\ln(n)}$ $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}3^n}{(2n+1)!}$ $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\,2^n}$

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n+1}$ $\sum_{n=1}^{\infty}\frac{3n(-1)^{n-1}}{n^2+3}$ $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n^2-3}$

f) $\sum_{n=1}^{\infty}\frac{(-10)^n}{4^{2n+1}(n+1)}$, $\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)!}$, $\sum_{n=1}^{\infty}\frac{(-1)^n}{(n^2+1)}$, $\sum_{n=1}^{\infty}\frac{n+2}{2n+7}$,

$\sum_{n=1}^{\infty}\frac{n^n}{3^{(1+2n)}}$, $\sum_{n=1}^{\infty}[\frac{5n-3n^3}{7n^3+2}]^n$, $\sum_{n=1}^{\infty}\frac{1}{3^n+n}$, $\sum_{n=1}^{\infty}\frac{n}{n^2-\cos^2(n)}$,

$\sum_{n=1}^{\infty}\frac{n^2+2}{n^4+5}$, $\sum_{n=1}^{\infty}\frac{4n^2+n}{\sqrt[3]{n^7+n^3}}$, $\sum_{n=0}^{\infty}\frac{(-1)^{n-3}\sqrt{n}}{n+4}$, $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{\sqrt{n}}$,

$\sum_{n=1}^{\infty}[\frac{2n^2+1}{n^2+1}]^n$, $\sum_{n=1}^{\infty}\frac{7}{3n^2+2n}$, $\sum_{n=2}^{\infty}\frac{1}{n\,\mathrm{Ln}(n)}$, $\sum_{n=1}^{\infty}n\,e^{n^2}$

$$\sum_{n=1}^{\infty}\frac{5^n+7^n}{3^n+2^n}, \qquad \sum_{n=2}^{\infty}\frac{Ln(n)}{n}, \qquad \sum_{n=1}^{\infty}\frac{(-1)^n}{3^n(2n!)^2}, \qquad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\,2^n},$$

II) Find interval of convergence for the following series and determine the behavior of the series at the endpoints of the interval. State clearly where the series converges absolutely, where it converges conditionally, and where it diverges.

$$\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}, \qquad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-2)^n}{n\,2^n}, \qquad \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!\,x^{2n+1}},$$

$$\sum_{n=1}^{\infty}n^3x^{2n}, \qquad \sum_{n=1}^{\infty}\frac{(x-3)^n}{n!}, \qquad \sum_{n=1}^{\infty}\frac{(-1)^n(x-6)^n}{n\,3^n}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{3^n(n!)^2} \qquad \sum_{n=1}^{\infty}\frac{(-1)^n(x-1)^{2n}}{2n!} \qquad \sum_{n=1}^{\infty}\frac{x^n}{n!}$$

$$\sum_{n=1}^{\infty}\frac{(x-2)^n}{n\,5^n} \qquad \sum_{n=1}^{\infty}n^nx^n \qquad \sum_{n=1}^{\infty}\frac{n(x+2)^n}{4^{n+1}}$$